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SINGULARITY OF SELF-SIMILAR:
MEASURES WITH RESPECT TO
HAUSDORFF MEASURES.

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SINGULARITY OF SELF-SIMILAR MEASURES WITH RESPECT TO HAUSDORFF MEASURES.

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Abstract

Besicovitch (1941) and Egglestone (1949) analyzed subsets of points of the unit interval with given frequencies in the figures of their base- p expansions. We extend this analysis to self-similar sets, by replacing the frequencies of figures with the frequencies of the generating similitudes. We focus on the interplay among such sets, self-similar measures, and Hausdorff measures. We give a fine-tuned classification of the Hausdorff measures according to the singularity of the self-similar measures with respect to those measures. We show that the self-similar measures are concentrated on sets whose frequencies of similitudes obey the law of the iterated logarithm.

1 Introduction

The research developed in this article is motivated by the following well-known example. Let $I = [0, 1) \subset \mathbb{R}$, and take an integer $m > 1$. Let $M = \{0, 1, \dots, m-1\}$. For $x \in I$ and $j \in M$, let $\delta_j(x)$ be the asymptotic frequency of the figure j in the non-terminating base- m expansion of x , $x = \sum_{i \in \mathbb{N}} a_i m^{-i}$ with $a_i \in M$. Given a set of positive numbers $\{p_0, \dots, p_{m-1}\}$ such that $\sum_{i \in M} p_i = 1$, consider the set $\Lambda_m(p_0, \dots, p_{m-1}) \subset I$ composed of those x such that $\delta_j(x) = p_j$ for $j \in M$.

In 1934 Besicovitch proved that the Hausdorff dimension of the set $\Lambda_2(p, 1-p)$ is given by $-\log_2(p^p(1-p)^{1-p})$ [Bes]. Egglestone generalized this formula for $m > 2$. More precisely, he proved [Egg] that the Hausdorff dimension of $\Lambda_m(p_0, \dots, p_{m-1})$ is given by

$$\dim \Lambda_m(p_0, \dots, p_{m-1}) = \frac{\sum_{i=0}^{m-1} p_i \log p_i}{-\log m}. \quad (1)$$

In 1960 Billingsley [Bil] considered Hausdorff measures on a probability space $(\Omega, \mathcal{A}, \mu)$ carrying a stochastic process $\{x_1, x_2, \dots\}$ with a finite state space. He found the Hausdorff dimension of sets of points characterized by the asymptotic frequency of the transitions between the states when the process $\{x_1, x_2, \dots\}$ is a regular Markov chain. The probability space $(\Omega, \mathcal{A}, \mu)$ can be chosen to be the unit interval in the real line with the Lebesgue measure, and then the results of Besicovitch and Egglestone turn out to be a particular case of those of Billingsley.

Iterated function systems with probabilities [Bar] can be regarded as geometrical realizations of the scheme of Billingsley. We introduce in this paper a natural extension of the definition of the Besicovitch-Egglestone sets in this context, as sets of points with given asymptotic frequencies of their generating contractions. In the case when the contractions are similitudes satisfying a suitable separation condition, the formula (1) for the dimension of the Besicovitch-Egglestone sets can be generalized in a natural way. This can be derived from results in [DGSH] and theorem 1 in this paper (see section 2).

Self-similar measures [Hut] can be considered as geometric projections of product probability measures from an abstract Cantor set. In this paper we analyze the relationships among such measures, Hausdorff measures, and certain sets that, as the Besicovitch-Egglestone sets, are characterized by the asymptotic properties of their generating similitudes. In section 2 we describe the main results of the paper and we give the proofs in section 3.

2 Main Results

Let $\Psi = \{\varphi_i : i \in M\}$, with $M = \{1, 2, \dots, m\}$, be a system of contractive similitudes of \mathbb{R}^N , where N is a positive integer. We call $\mathcal{S}(N, M)$ the set of such systems. There exists a unique compact set E such that $E = S\Psi(E) := \bigcup_{i \in M} \varphi_i(E)$ [Hut]. E is called the self-similar set generated by Ψ . There exists a surjective mapping π from the product space $M^\infty := M \times M \times \dots$ onto E given by

$$\pi(i) = \bigcap_{k \in \mathbb{N}} (\varphi_{i_1} \circ \varphi_{i_2} \circ \dots \circ \varphi_{i_k}(E)), \quad \text{for } i = (i_1 i_2 \dots) \in M^\infty.$$

Let s be the unique real number such that $\sum_{i \in M} r_i^s = 1$, where r_i is the contraction ratio of φ_i for $i \in M$. We assume that the standard *open set condition* (OSC) holds for the system Ψ (see section 3). It is known that the Hausdorff dimension of E is given by s [Hut].

Let \mathcal{P}^+ be the set of positive probability vectors in M , that is, $\mathbf{p} \in \mathcal{P}^+$ iff $\mathbf{p} = (p_i : i \in M)$ with $p_i > 0$ for all i and $\sum_{i \in M} p_i = 1$. Given $\mathbf{p} \in \mathcal{P}^+$, we denote by $\nu_{\mathbf{p}}$ the probability measure on M^∞ defined by the infinite-fold product measure $\mathbf{p} \times \mathbf{p} \times \dots$. Let $\mu_{\mathbf{p}}$ be the projected product measure supported by E , that is $\mu_{\mathbf{p}} = \nu_{\mathbf{p}} \circ \pi^{-1}$. Let $\mathcal{M}^+ = \{\mu_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}^+\}$ be the set of such measures on E , also called *self-similar measures* [Ban]. We will write \mathbf{p}_s for the choice $\mathbf{p} = (r_i^s : i \in M)$. In [DGSH] it was proved that the Hausdorff dimension of the measure $\mu_{\mathbf{p}}$ is given by

$$\dim \mu_{\mathbf{p}} = s(\mathbf{p}) := \frac{\sum_{i \in M} p_i \log p_i}{\sum_{i \in M} p_i \log r_i}, \quad (2)$$

where $\dim \mu = \inf\{\dim A : \mu(A) = 1\}$ and $\dim(\cdot)$ stands for the Hausdorff dimension (see section 3 for a definition). Observe that $s(\mathbf{p}) = s$ for $\mathbf{p} = \mathbf{p}_s$.

We now formulate the main results in this paper. Let Θ be the *overlapping set* of E , that is

$$\Theta = \bigcup_{i \neq j} (\varphi_i(E) \cap \varphi_j(E)). \quad (3)$$

The following theorem will be used throughout the paper.

Theorem 1. (Θ -lemma)

$$\mu(\Theta) = 0 \text{ for all } \mu \in \mathcal{M}^+.$$

Theorem 1 solves a problem posed by C. Bandt [Ban] about the singularity of self-similar measures when π is not injective. It is well-known that different choices of \mathbf{p} yield measures $\nu_{\mathbf{p}}$ on M^∞ which are mutually singular (see e.g. [Wal]).

This obviously implies that the induced self-similar measures μ_p are also mutually singular when π is a bijection. Whether the same property holds when π fails to be an injection is an open problem [Ban]. The result above shows that, under the OSC, two distinct self-similar measures are mutually singular.

Let H^ϕ denote the ϕ -Hausdorff measure associated with a dimension function ϕ (see section 3 for a definition). We write H^t for $\phi(\xi) = \xi^t$. We now state the main result in this paper, which concerns the geometry of self-similar measures. Observe that it cannot be decided from (2) whether the measure μ_p is singular or absolutely continuous with respect to the Hausdorff measure $H^{s(p)}$. The following theorem can be proved, which answers this problem by means of a fine-tuned classification of Hausdorff measures.

Theorem 2.

Let $\Psi \in \mathcal{S}(N, M)$ satisfying the OSC and $p \in \mathcal{P}^+$ be given. Let $\alpha \geq 0$ and $\mathcal{G} = \{\phi_\alpha\}_{\alpha \geq 0}$ be the one-parameter family of real functions

$$\phi_\alpha(\xi) = \xi^{s(p)} \exp\{\alpha(2 \log \xi^{c(p)} \log \log \log \xi^{c(p)})^{1/2}\}, \quad (4)$$

where $s(p)$ is given in (2), and

$$c(p) = \left(\sum_{i \in M} r_i \log p_i\right)^{-1} < 0. \quad (5)$$

Let

$$d(p) = \left(\sum_{i \in M} (\log p_i - s(p) \log r_i)^2 p_i\right)^{1/2}. \quad (6)$$

Then, for the self-similar measure μ_p induced by p , it holds

- i) μ_p is singular w.r.t. H^{ϕ_α} for $0 \leq \alpha < d(p)$.
- ii) μ_p is absolutely continuous w.r.t. H^{ϕ_α} for $\alpha > d(p)$.
- iii) μ_p has an integral representation w.r.t. a Hausdorff measure H^t if and only if $p = p_s$, and thus $t = s$.

Since $H^{s(p)} \leq H^{\phi_\alpha}$ for all $\alpha > 0$, theorem 2 implies that, for $p \neq p_s$, the self-similar measure μ_p is singular with respect to the Hausdorff measure $H^{s(p)}$. On the other hand, theorem 2 allows us to discern which of two given measures with the same dimension is concentrated on a “smaller” set (in the sense of the ϕ -Hausdorff measure). Notice that $s(p) = s(q)$ for $p \neq q$ is a plausible case. In such a case, the discriminating parameter is the standard deviation $d(p)$ of a random variable which depends on the pair (Ψ, μ_p) . More precisely, if $s(p) = s(q)$ for $p \neq q$ but $d(p) > d(q)$, then $H^{\phi_\alpha}(B) = +\infty$ for any set B of positive μ_q -measure and for

all $d(p) > \alpha > d(q)$, whereas $H^{\phi_\alpha}(D) = 0$ for a set D of full μ_p -measure. Thus, if $d(p) > d(q)$, the measure μ_p is concentrated on a smaller set (from a geometric point of view) than that on which μ_q is.

We next turn our attention to some natural sets that concentrate the measure μ_p thus providing an alternative approach to self-similar geometry to that supplied by self-similar measures.

For $j \in M$ and $i = (i_1, i_2, \dots) \in M^\infty$, let $\delta_j(i)$ be the asymptotic frequency of the symbol j in the sequence i , i.e.

$$\delta_j(i) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{card}\{k: i_k = j, 1 \leq k \leq n\}.$$

We call the *Besicovitch set* of $\Psi \in \mathcal{S}(N, M)$ associated with the probability vector $p = (p_i)_{i \in M}$ the subset of E given by

$$B_p = \{\pi(i): i \in M^\infty, \delta_j(i) = p_j \text{ for all } j \in M\}. \quad (7)$$

The Hausdorff dimension of the sets B_p is shown to be $s(p)$. This is in essence a consequence of the law of large numbers, formula (2), and the Θ -lemma. In particular, since $\mu_p(B_p) = 1$ because of the strong law of large numbers, (2) provides a lower bound for the dimension of B_p . An argument involving suitable coverings of the set B_p could be used to prove that $\dim B_p = s(p)$, provided that the overlapping set Θ has zero μ_p -measure (which is the content of theorem 1).

For $j \in M^k := M \times \dots \times M$, let $p_j = p_{j_1} p_{j_2} \dots p_{j_k}$, and let $\delta_j(i)$ denote the asymptotic frequency of appearance of the finite sequence j in the infinite code $i \in M^\infty$, i.e.

$$\delta_j(i) = \lim_{n \rightarrow +\infty} \frac{1}{n} \text{card}\{q: i_q = j_1, i_{q+1} = j_2, \dots, i_{q+k-1} = j_k, 1 \leq q \leq n\}. \quad (8)$$

We also define the *thin Besicovitch set* $B_p^{(\infty)}$ of $\Psi \in \mathcal{S}(N, M)$ associated with p by

$$B_p^{(\infty)} = \bigcap_{k \in \mathbb{N}} B_p^{(k)}, \quad (9)$$

where

$$B_p^{(k)} = \{\pi(i): i \in M^\infty, \delta_j(i) = p_j, \text{ for all } j \in M^k\}. \quad (10)$$

Birkhoff's ergodic theorem implies that $\mu_p(B_p^{(\infty)}) = 1$, and thus $\dim B_p^{(\infty)} \geq s(p)$ from (2). Since $B_p^{(\infty)} \subset B_p$, then $\dim B_p^{(\infty)} = s(p)$.

Let τ be the shift mapping on M^∞ , i.e.

$$\tau(i_1, i_2, i_3, \dots) = (i_2, i_3, \dots). \quad (11)$$

Given a random variable $Z: M \mapsto \mathbb{R}$ and a sequence $\mathbf{i} = (i_1, i_2, \dots) \in M^\infty$, let

$$S_n^Z(\mathbf{i}) = \sum_{j=1}^n Z \circ \text{pr}_1 \circ \tau^j(\mathbf{i}), \quad (12)$$

where pr_1 is the projection mapping $M^\infty \mapsto M$ onto the first coordinate

$$\text{pr}_1(i_1, i_2, \dots) = i_1. \quad (13)$$

Let X be the random variable $M \mapsto \mathbb{R}$ defined by

$$X(i) = \log p_i - s(\mathbf{p}) \log r_i, \quad i \in M. \quad (14)$$

Let $\mathcal{E}[\cdot]$ denote the expectation, with respect to the probability \mathbf{p} , of a random variable defined in M . Observe that $\mathcal{E}[X] = 0$ and $\mathcal{E}[X^2] = d(\mathbf{p})^2$, where $d(\mathbf{p})$ is defined in (6). Consider the set $L_{\mathbf{p}}$ defined by

$$L_{\mathbf{p}} = \{\pi(\mathbf{i}) \in B_{\mathbf{p}} : \mathbf{i} \in M^\infty, \quad \limsup_n ((2n \log \log n)^{-1/2} S_n^X(\mathbf{i})) = d(\mathbf{p}), \quad (15)$$

$$\liminf_n ((2n \log \log n)^{-1/2} S_n^X(\mathbf{i})) = -d(\mathbf{p})\}.$$

The set $L_{\mathbf{p}}$ carries the whole $\mu_{\mathbf{p}}$ -measure, because of the law of the iterated logarithm. This permits to show that $\dim L_{\mathbf{p}} = s(\mathbf{p})$, since $L_{\mathbf{p}} \subset B_{\mathbf{p}}$. Furthermore, the set $L_{\mathbf{p}}$ characterizes a null $H^{s(\mathbf{p})}$ -measure set concentrating $\mu_{\mathbf{p}}$, which is guaranteed to exist by virtue of theorem 2 part i).

We collect the above facts, as well as others concerning the Hausdorff measure, in the

Theorem 3.

Let $\mathbf{p} \in \mathcal{P}^+$. Let $B_{\mathbf{p}}$, $B_{\mathbf{p}}^{(\infty)}$, and $L_{\mathbf{p}}$ be the sets defined in (7), (9), and (16) respectively. Then

i) $\dim B_{\mathbf{p}} = \dim B_{\mathbf{p}}^{(\infty)} = s(\mathbf{p})$.

ii) $0 < H^s(B_{\mathbf{p}_s}) = H^s(B_{\mathbf{p}_s}^{(\infty)}) = H^s(L_{\mathbf{p}_s}) = H^s(E) < +\infty$.

iii) For any $\mathbf{p} \neq \mathbf{p}_s$, the $H^{s(\mathbf{p})}$ -measure of the set $B_{\mathbf{p}}^{(\infty)}$ is either zero or infinity. If $\dim(B_{\mathbf{p}} \cap \Theta) < s(\mathbf{p})$, then the $H^{s(\mathbf{p})}$ -measure of the set $B_{\mathbf{p}}$ is either zero or infinity.

iv) Let $\mathbf{p} \neq \mathbf{p}_s$. Then $L_{\mathbf{p}}$ is a set of full $\mu_{\mathbf{p}}$ -measure, and Hausdorff dimension $s(\mathbf{p})$. Furthermore, $H^{s(\mathbf{p})}(L_{\mathbf{p}}) = 0$.

Part ii) of theorem 3 provides a good reason to regard the points of $B_{\mathbf{p}_s}$ as the "normal" points (in the sense of Borel) of the self-similar set E . Moreover, $B_{\mathbf{p}_s}^{(\infty)}$ can

be thought of as the true "core" of the set E . Part iii) is a consequence of a stronger result on $S\Psi$ -invariant sets (see proposition 3.7). It follows from that result that there is no $S\Psi$ -invariant t -set (that is, with finite and positive H^t -measure) in E for $\dim\Theta < t < s$.

The problem of finding the exact Hausdorff measure of the Besicovitch sets in their dimension remains open. Part iii) in the above theorem gives some clues about this question.

In order to see that Besicovitch-Egglestone sets are obtained as a particular case of our scheme, let $M = \{0, 1, \dots, m-1\}$, and consider the system of similitudes $\Psi = \{\varphi_i : i \in M\}$ on $I=[0,1]$, where $\varphi_i(x) = m^{-1}(x+i)$ for $i \in M$. Then I is the self-similar set generated by Ψ , and $\Lambda_m(p_0, \dots, p_{m-1})$ is the Besicovitch set B_p of Ψ associated with $p = (p_0, \dots, p_{m-1})$ as defined in (7). The Hausdorff dimension of $\Lambda_m(p_0, \dots, p_{m-1})$ given in (1) is obtained from (2) by substituting $r_i = m^{-1}$ for every i . Therefore (2) generalizes formula (1) to self-similar sets. Moreover theorem 3 applies to the sets $\Lambda_m(p_0, \dots, p_{m-1})$. In particular, the sets of Besicovitch and Egglestone have either zero or infinite Hausdorff measure in their dimension.

Remark 4

We remark here that the dimension formula in (2) can be interpreted in terms of the ergodic theory of dynamical systems. More in particular, we pursue an explanation of $s(p)$ in the spirit of [You]. See [Wal] for definitions.

Let $p \in \mathcal{P}^+$ and consider the dynamical system (E, T, μ_p) , where T is the shift mapping on the self-similar set; that is, T is defined by $T \circ \pi = \pi \circ \tau$, τ being the Bernoulli shift on M^∞ defined in (11). From the Θ -lemma we learn that the measure spaces (M^∞, ν_p) and (E, μ_p) are isomorphic (both endowed with the σ -algebra generated by the class of cylinders, see remark 3.2). Moreover, T and τ are isomorphic measure-preserving transformations. Therefore μ_p and ν_p has the same *measure-theoretic entropy*, given by $h(\mu_p) = -\sum_{i \in M} p_i \log p_i$ (see [Wal]).

The Raghunathan's version of Oseledec's theorem [Rag] allows the computation of the whole Liapunov spectrum of the system (E, T, μ_p) . Specifically, using the law of large numbers, it can be proved that all the N *Liapunov exponents* of the system are given by

$$\lambda(\Psi, \mu_p) := -\sum_{i \in M} p_i \log r_i > 0$$

for μ_p -almost all $x \in E$ (which in particular implies that the dynamical system (E, T, μ_p) displays sensitive dependence on initial conditions). Taking into account

these two facts, the formula (2) for $s(p)$ can be written as

$$\dim \mu_p = \frac{h(\mu_p)}{\lambda(\Psi, \mu_p)}. \quad (16)$$

Observe that the right-hand side of formula (16) is always defined for any $\mu_p \in \mathcal{M}^+$.

3 Proofs

We first give some basic definitions and notation.

Given $A \subset \mathbb{R}^N$ and $\delta > 0$, a δ -covering of the set A is a collection of sets $\{U_i\}_{i \in \mathbb{N}}$ such that $\bigcup_i U_i \supset A$ with $|U_i| \leq \delta$, where $|\cdot|$ stands for the diameter of a set.

Let \mathcal{F} denote the set of dimension functions, defined by

$$\mathcal{F} = \{\phi: (0, \delta) \mapsto \mathbb{R}^+ / \phi \text{ continuous, increasing, } \lim_{\xi \rightarrow 0^+} \phi(\xi) = 0, 0 < \delta < 1\}$$

We will use the spherical ϕ -Hausdorff measure throughout this paper, i.e. for $\phi \in \mathcal{F}$, let

$$H^\phi(A) = \sup_{\delta > 0} \inf \left\{ \sum_{i \in \mathbb{N}} \phi(|B_i|) : \{B_i\} \text{ is a } \delta\text{-covering of } A \text{ by balls} \right\}.$$

The expression behind the supremum is denoted by $H_\delta^\phi(A)$. For $\phi \in \mathcal{F}$ satisfying $\limsup_{\xi \rightarrow 0} (\phi(2\xi)/\phi(\xi)) = \phi^* < +\infty$, H^ϕ is comparable to the standard ϕ -Hausdorff measure H^ϕ , in the sense that $(\phi^*)^{-1} H^\phi(A) \leq H^\phi(A) \leq H^\phi(A)$ for $A \subset \mathbb{R}^N$.

Let H^a be the Hausdorff measure associated with the dimension function $\phi(\xi) = \xi^a$. The Hausdorff dimension $\dim A$ of a subset $A \subset \mathbb{R}^N$ is given by the threshold value

$$\dim A = \sup\{a: H^a(A) > 0\} = \inf\{a: H^a(A) < +\infty\}.$$

We call U and u , respectively, the maximum and minimum of the set $\{r_i: i \in M\}$ (r_i is the contraction ratio of the similitude φ_i). We assume that $U < 1$.

We say that Ψ fulfills the open set condition if there exists an open set $V \subset \mathbb{R}^N$ such that $S\Psi(V) \subseteq V$ and $\varphi_i(V) \cap \varphi_j(V) = \emptyset$ for $i, j \in M$ with $i \neq j$. We write \bar{V} for the closure of V , \bar{V} . We may assume without loss of generality that $|V| = 1$.

We write M^* for the set of finite sequences with terms in M . Given a sequence $j = (j_1, j_2, \dots, j_k) \in M^k$, φ_j stands for the composite similitude $\varphi_{j_1} \circ \varphi_{j_2} \circ \dots \circ \varphi_{j_k}$, and r_j does for its contraction ratio $r_{j_1} r_{j_2} \dots r_{j_k}$. A *cylinder* is the set of sequences of M^∞ whose k -first figures are those of j , that is

$$\{(j_1, j_2, \dots, j_k, i_{k+1}, i_{k+2}, \dots) : i_n \in M, n > k\}. \quad (17)$$

For $k \in \mathbb{N}$ and $\mathbf{i} \in M^\infty$, $\mathbf{i}(k)$ denotes the curtailed sequence $(i_1 i_2 \dots i_k)$.

E_j and F_j stand respectively for the image sets $\varphi_j(E)$ and $\varphi_j(F)$, which will also be called cylinders (on E).

Observe that the set

$$\Theta^* = \{x \in E : x = \pi(\mathbf{i}) = \pi(\mathbf{j}) \text{ for some pair } \mathbf{i}, \mathbf{j} \in M^\infty, \mathbf{i} \neq \mathbf{j}\}$$

can be written as the countable union $\Theta^* = \bigcup_{\mathbf{i} \in M^\infty} \varphi_{\mathbf{i}}(\Theta)$. In particular this implies that, for any $\mu \in \mathcal{M}^+$, if Θ is a μ -null set then Θ^* also is. We first give the proof of the Θ -lemma.

Proof of theorem 1. Since V satisfies the OSC for the system Ψ , it is known [Sch] that $V \cap E \neq \emptyset$ (strong OSC) holds when M is finite.

Given an $x \in E$ and an $\mathbf{i} \in M^\infty$ such that $\pi(\mathbf{i}) = x$ we define the shift \mathbf{i} -orbit of x as the set $\gamma_{\mathbf{i}}(x) = \{x, x_1, x_2, \dots\}$, where for $m \geq 1$, $x_m = \varphi_{i_m}^{-1} \circ \dots \circ \varphi_{i_1}^{-1}(x)$, and we define the shift orbit of x as $O(x) := \bigcup \{\gamma_{\mathbf{i}} : \mathbf{i} \in M^\infty, \pi(\mathbf{i}) = x\}$.

For $\mathbf{j} \in M^*$, let $\delta_{\mathbf{j}}(\mathbf{i})$ be the limit in (8), and consider the ‘thin’ Besicovitch subset of the space M^∞

$$\mathcal{B}_p^{(\infty)} = \{\mathbf{i} \in M^\infty : \delta_{\mathbf{j}}(\mathbf{i}) = p_{\mathbf{j}}, \mathbf{j} \in M^*\}$$

(recall that $p_{\mathbf{j}} = p_{j_1} p_{j_2} \dots p_{j_k}$ for $\mathbf{j} \in M^k$).

Observe the following two facts.

(i) Since $p \in \mathcal{P}^+$, $O(\pi(\mathbf{i}))$ is dense in E for each $\mathbf{i} \in \mathcal{B}_p^{(\infty)}$.

(ii) $O(x) \subset \partial V$ for all $x \in \partial V \cap E$, because otherwise, $x_m \in \gamma_{\mathbf{i}}(x) \cap V$ for some $m \in \mathbb{N}$ and for some $\mathbf{i} \in M^\infty$, would imply that $x_{m-1} = \varphi_{i_m}(x_m) \in \gamma_{\mathbf{i}}(x) \cap V$, and recursively that $x \in V$.

Assume there exists an $x \in \pi(\mathcal{B}_p^{(\infty)}) \cap \Theta$. Since $x \in \Theta$, $x = \pi(\mathbf{i}) = \pi(\mathbf{j})$ for $\mathbf{i}, \mathbf{j} \in M^\infty$ and $x \in \varphi_{i_1}(E) \cap \varphi_{j_1}(E) \subset \varphi_{i_1}(F) \cap \varphi_{j_1}(F)$ with $i_1 \neq j_1$. Moreover, from the OSC, $x \in \varphi_{i_1}(\partial V) \cap \varphi_{j_1}(\partial V)$. Therefore, if $\gamma_{\mathbf{i}}(x) = \{x, x_1, x_2, \dots\}$, then $x_1 \in \partial V \cap E$, and $\gamma_{i_2 i_3 \dots}(x_1) \subset \partial V$ from (ii). Since $\gamma_{i_2 i_3 \dots}(x_1)$ is dense in E , ∂V is also dense in E which contradicts the strong OSC. Hence $\pi(\mathcal{B}_p^{(\infty)}) \cap \Theta = \emptyset$, and in particular $\mu_p(\Theta) = 0$ since $\mu_p(\pi(\mathcal{B}_p^{(\infty)})) = \nu_p(\mathcal{B}_p^{(\infty)}) = 1$ from the Birkhoff’s ergodic theorem. \square

Remark 3.1 Assume $M = \mathbb{N}$. Even though we are not concerned in this paper with self-similar sets generated by an infinite system of similitudes fulfilling OSC

[Mor] , it is worth mentioning that theorem 1 also holds in this case. We give here a justification. If \mathbb{I} is any finite subset of \mathbb{N} , and we consider the system $\Psi_{\mathbb{I}} = \{\varphi_i : i \in \mathbb{I}\}$ with self-similar set $E_{\mathbb{I}}$, V obviously fulfills the OSC for $\Psi_{\mathbb{I}}$. From [Sch], $V \cap E_{\mathbb{I}} \neq \emptyset$ and therefore $V \cap E \neq \emptyset$. Thus the strong OSC holds, and the proof of theorem 1 also applies to the infinite case.

Remark 3.2 Let $\Sigma^\infty = \pi^{-1}(E \setminus \Theta^*)$. The mapping $\pi|_{\Sigma^\infty}$ is a bijection onto $E \setminus \Theta^*$. This allows us to consider a unique fixed $i_x = \pi^{-1}(x) \in \Sigma^\infty$ for each $x \in E \setminus \Theta^*$. We will use this fact often throughout this section. From the Θ -lemma $\mu_p(E \setminus \Theta^*) = 1$ for every positive probability p on M . More precisely, the Θ -lemma implies that, for each $p \in \mathcal{P}^+$, the measure spaces (M^∞, C, ν_p) and (E, C_π, μ_p) are isomorphic, where C_π denotes the σ -algebra on E induced by π from the σ -algebra C generated by the class of cylinders on M^∞ .

From now onwards, let p be any fixed probability in \mathcal{P}^+ . For convenience, we drop the subindex p from μ_p . We need some previous results in order to prove theorems 2 and 3.

Given a product measure $\mu \in \mathcal{M}^+$ and a dimension function $\phi \in \mathcal{F}$ we define the following ϕ -upper density of μ over (geometrical) cylinders at $x \in E$

$$\bar{d}_\mu^\phi(x) = \sup \left\{ \limsup_{k \rightarrow +\infty} \frac{\mu(F_{i(k)})}{\phi(r_{i(k)})} : i \in \pi^{-1}(x) \right\} \quad (18)$$

We will write \bar{d}_μ^t when $\phi(\xi) = \xi^t$.

For each ball B of diameter $r > 0$, consider the collection of sets $G(B)$ given by

$$G(B) = \{F_{i(k)} : i \in M^\infty, F_{i(k)} \cap B \neq \emptyset, r_{i(k)} \leq r \text{ and } r_{i(k-1)} > r\} \quad (19)$$

The following lemma is a slightly extended version of a well-known one [MM].

Lemma 3.3 *Let B be any open ball with diameter $r > 0$, then*

$$\text{i) } \text{card} G(B) \leq q < +\infty \text{ independently of } r. \quad (20)$$

$$\text{ii) } \sum_{P \in G(B)} \phi(|P|) \leq q\phi(r) \text{ for any } \phi \in \mathcal{F} \quad (21)$$

We say that a family of sets $\{U_i\}_{i \in \mathbb{N}} \subset E$ is μ -disjointed if $\mu(U_i \cap U_j) = 0$ for $i \neq j$. From the Θ -lemma, every covering of a subset of E by (geometric) cylinders can be taken to be μ -disjointed.

Lemma 3.4 Let $\mathcal{U} = \{C_i\}_{i \in \mathbb{N}}$ be a family of cylinders in E , i.e. $\mathcal{U} \subset \{E_i : i \in M^*\}$, such that $C_i \not\subset C_j$ for $i \neq j$. Then \mathcal{U} is μ -disjointed.

Theorem 3.5 Let $A \subset E$, and a, b be two positive constants.

- i) If $\sup_{x \in A} \bar{d}_\mu^\phi(x) < a$, then $H^\phi(A) \geq (aq)^{-1} \mu(A)$, where q is as in (20).
- ii) If $\inf_{x \in A} \bar{d}_\mu^\phi(x) > b$, then $H^\phi(A) \leq b^{-1}$.

proof:

- i) Let $n \in \mathbb{N}$, and consider the set

$$A_n = \{\pi(i) \in A \setminus \Theta^* : i \in M^\infty, \text{ for all } k \geq n \mu(F_{i(k)}) < (a-\varepsilon)\phi(r_{i(k)}) \text{ for some } \varepsilon > 0\}.$$

Then $\{A_n\}$ is a non-decreasing sequence of sets with $\bigcup_n A_n \subseteq A$ and such that $\mu(A \setminus \bigcup_n A_n) = 0$, because of the Θ -lemma. Consider a countable δ -covering $\mathcal{R}_n = \{B_j\}_j$ of A_n by balls, with $u^n > \delta$, and take, for each $B_j \in \mathcal{R}_n$, the collection

$$G_j(n) = \{P \in G(B_j) : P \cap A_n \neq \emptyset\},$$

where $G(B_j)$ is defined in (19). Now, for the chosen covering, we obtain

$$\begin{aligned} \sum_{B_j \in \mathcal{R}_n} \phi(|B_j|) &\geq q^{-1} \sum_j \sum_{P \in G_j(n)} \phi(|P|) \geq \\ &\geq q^{-1} a^{-1} \sum_j \sum_{G_j(n)} \mu(P) \geq (qa)^{-1} \mu(A_n), \end{aligned}$$

where (21), the definition of $G_j(n)$, and the fact that $\bigcup_j \bigcup_{G_j(n)} P \supset A_n$ have been applied in that order. Thus we obtain that $H_\delta^\phi(A) \geq H_\delta^\phi(A_n) \geq (aq)^{-1} \mu(A_n)$ for each n . Letting δ tend to 0, we obtain $H^\phi(A) \geq (aq)^{-1} \mu(A)$, since $\mu(A_n)$ increases to $\mu(A)$.

- ii) Let $\varepsilon > 0$ with $b - \varepsilon > 0$. For each $x \in A$, choose some $i_x \in \pi^{-1}(x)$ such that the inequality $\mu(F_{i_x(k)}) > (b - \varepsilon)\phi(r_{i_x(k)})$ holds for infinitely many k . Now, for $\delta > 0$, take $K \in \mathbb{N}$ such that $U^K < \delta$. For each $x \in A$ consider the integer

$$k(x) = \min\{k : k > K, \mu(F_{i_x(k)}) > (b - \varepsilon)\phi(r_{i_x(k)})\}.$$

The collection $\{F_{i_x(k)} : k = k(x), x \in A\}$ can be taken to be a μ -disjointed δ -covering, \mathcal{U} , of A by cylinders. Therefore

$$\sum_{\mathcal{U}} \phi(r_{i(k)}) \leq (b - \varepsilon)^{-1} \sum_{\mathcal{U}} \mu(F_{i(k)}) \leq (b - \varepsilon)^{-1} \mu(E),$$

so that $H_\delta^\phi \leq (b - \varepsilon)^{-1}$ for an $\varepsilon > 0$ arbitrarily small, and the result follows. \square

Lemma 3.6 Let $c = c(p)$ as given in (5), and $f_c(\xi) = (2 \log \xi^c \log \log \log \xi^c)^{1/2}$. Then, for every $i \in M^\infty$ such that $\pi(i) \in B_p$

$$\lim_{k \rightarrow +\infty} \frac{f_c(r_{1(k)})}{(2k \log \log k)^{1/2}} = 1 \quad (22)$$

proof:

Let R be the random variable on M defined by $R(i) = \log r_i$, $i \in M$. Consider the Bernoulli process $\{R_j\}_j$, defined by $R_j = R \circ \text{pr}_1 \circ \tau^j$, where pr_1 is the projection onto the first coordinate (see (13)) and τ is the shift mapping in (11). Observe that $S_k^R(i) = \log r_{1(k)}$, where the notation in (12) is used.

Let $i \in M^\infty$ such that $\pi(i) \in B_p$. It can be seen that $\lim_{k \rightarrow \infty} k^{-1} S_k^R(i) = c^{-1}$. Let $0 < \varepsilon < c^{-1}$, and take k_1 such that $k(1 + c\varepsilon) < c \log r_{1(k)} < k(1 - c\varepsilon)$ holds for $k > k_1$. Let k_2 be large enough so that both $\log \log(1 + c\varepsilon)k / \log \log k > 1 + c\varepsilon$ and $\log \log(1 - c\varepsilon)k / \log \log k < 1 - c\varepsilon$ hold for $k > k_2$. Now, for $k > \max\{k_1, k_2\}$, the inequalities

$$1 + c\varepsilon < \frac{f_c(r_{1(k)})}{(2k \log \log k)^{1/2}} < 1 - c\varepsilon$$

hold, and thus (22) follows. \square



Proof of theorem 2. Given a Borel measure μ , C.A. Rogers and S.J. Taylor [RT] characterized those ϕ -Hausdorff measures with respect to (w.r.t.) which μ is singular and those with respect to which μ is absolutely continuous in terms of the standard ϕ -upper density (e.g. over dyadic intervals) of μ . The key result to obtain such a characterization in terms of the density (18) defined over cylinders is the density theorem 3.5. It allows us to use the above mentioned characterization by means of the following version of the theorem in [RT].

Rogers-Taylor Theorem

Let μ be a measure in \mathbb{R}^N with σ -algebra \mathcal{A} , and let $\phi \in \mathcal{F}$. Assume there exist an \mathcal{A} -measurable function $d = d_{(\mu, \phi)}$, $d: \mathbb{R}^N \mapsto [0, +\infty]$ and a constant $C > 0$ such that

- (i) If $d(x) < a$ for all $x \in A$, then $aCH^\phi(A) \geq \mu(A)$ for any $A \in \mathcal{A}$.
- (ii) If $d(x) > b$ for all $x \in A$, then $H^\phi(A) \leq Cb^{-1}$.

Then

- (a) μ is absolutely continuous w.r.t. H^ϕ if and only if $d(x) < +\infty$ μ -a.e.
- (b) μ is singular w.r.t. H^ϕ if and only if $d(x) = +\infty$ μ -a.e.

(c) μ has an integral representation w.r.t. H^ϕ if and only if $0 < d(x) < +\infty$ μ -a.e.

(A measure μ has an integral representation w.r.t. the measure H^ϕ if there exists a Borel set E_0 with ϕ -Hausdorff measure σ -finite and an H^ϕ -integrable function f such that $\mu(A) = \int_{A \cap E_0} f(x) dH^\phi$).

We now proceed to prove theorem 2. We take the upper density $\bar{d}_\mu^\phi(\cdot)$ defined in (18) as the function $d(\cdot)$ in the Rogers-Taylor theorem. Some calculus allows to check that ϕ_α actually belongs to the class \mathcal{F} . Notice that the measure-spaces isomorphism in remark 3.2 along with theorem 3.5 supply the hypothesis in the Rogers-Taylor theorem for the self-similar measure μ and the point function \bar{d}_μ^ϕ .

Let X be the random variable defined in (14). Recall that $\mathcal{E}[X] = 0$ and $\mathcal{E}[X^2] = d(p)^2$. Consider the Bernoulli process $\{X_k\}_{k \in \mathbb{N}}$, where $X_k = X \circ \text{pr}_1 \circ \tau^k$ (pr_1 and τ were defined in (13) and (11) respectively). Let \mathcal{G} be the family defined in (4). Note that the function $\phi_\alpha \in \mathcal{G}$ can be written as $\phi_\alpha(\xi) = \xi^{s(p)} \exp(\alpha f_{c(p)}(\xi))$, where $f_{c(p)}$ is as in lemma 3.6. For $\alpha > 0$, the ϕ_α -upper density at each $x \in E \setminus \Theta^*$ can be written as

$$\bar{d}_\mu^{\phi_\alpha}(x) = \exp\left\{\limsup_{n \rightarrow \infty} f_c(r_{l(n)}) \left(\frac{S_n^X(i)}{f_c(r_{l(n)})} - \alpha \right)\right\}, \quad (23)$$

where $i = \pi^{-1}(x)$.

From the Θ -lemma, lemma 3.6 and the law of the iterated logarithm applied to the sequence of random variables S_n^X , we get from expression (23) that for a set of full μ -measure $\bar{d}_\mu^{\phi_\alpha}(x) = +\infty$ if $0 \leq \alpha < d(p)$, and $\bar{d}_\mu^{\phi_\alpha}(x) = 0$ if $\alpha > d(p)$. Therefore, from parts (a) and (b) in the Rogers-Taylor theorem, parts i) and ii) of theorem 2 follow.

Consider now, for $t > 0$, the random variable $X^{(t)}$ defined by

$$X^{(t)}(i) = \log p_i - t \log r_i, \quad i \in M, \quad (24)$$

and observe that the t -upper density at $x \in E \setminus \Theta^*$ can be written as

$$\bar{d}_\mu^t(x) = \exp\left(\limsup_n (S_n^{X^{(t)}}(\pi^{-1}(x)))\right),$$

so that the boundedness of the density $0 < \bar{d}_\mu^t(x) < +\infty$ for μ -almost every x and for some $t > 0$ is equivalent to the fact that

$$-\infty < \limsup_n S_n^{X^{(t)}}(i) < +\infty$$

for ν -almost every i . This boundedness requires $\mathcal{E}[X^{(t)}] = 0$ and $\mathcal{E}[X^{(t)^2}] = 0$, and thus $t = s(p)$ and $p_i = r_i^{s(p)}$ for $i \in M$. This is only plausible if $s(p) = s$. Part (c) in the theorem by Rogers and Taylor proves part (iii) of theorem 2. \square

We now proceed to prove theorem 3. We firstly remark on some general facts about Besicovitch sets. Observe that a Besicovitch set can also be written as

$$B_p = \bigcap_{n \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{k \geq N} \bigcap_{j \in M} \{\pi(i) : i \in M^\infty \text{ and } p_j - n^{-1} < \delta_j(i, k) < p_j + n^{-1}\},$$

where the notation $\delta_j(i, k)$ stands for the frequency of the symbol j in the finite sequence $i(k)$. The bracketed set in that expression is a finite union of cylinders E_j , $j \in M^*$, so that it is a closed set. This shows that a Besicovitch set is an $F_{\sigma\delta}$ set, and as a consequence it is a Borel set. Moreover, the Besicovitch sets are non-compact self-similar ($S\Psi$ -invariant) sets, i.e. $S\Psi(B_p) = B_p$ for any p in M . Observe also that B_p is dense in E for any $p \in \mathcal{P}^+$. Notice that the same properties hold for the thin Besicovitch sets $B_p^{(\infty)}$ and the set L_p .

Proof of theorem 3 part i). Given (2) and the definitions (7) and (9), we only need to prove that $\dim B_p \leq s(p)$. This can be directly obtained from theorem 3.5. Let $t > s(p)$, and consider the random variable $X^{(t)}$ defined in (24). Notice that the t -upper density at any $x \in B_p$ verifies

$$\bar{d}_\mu^t(x) \geq \exp(\limsup_n S_n^{X^{(t)}}(i)) \quad (25)$$

for any $i \in \pi^{-1}(x)$. Since $\mathcal{E}[X^{(t)}] > 0$, and

$$\lim_n n^{-1} S_n^{X^{(t)}}(i_x) = \mathcal{E}[X^{(t)}]$$

for some $i_x \in \pi^{-1}(x)$, we get from (25) that $\bar{d}_\mu^t(x) = +\infty$ for all $x \in B_p$. From theorem 3.5 $H^t(B_p) = 0$, for B_p is a full μ -measure set. Since $t > s(p)$ is arbitrary, we have the desired inequality. \square

Proof of theorem 3 part ii). Let μ_s denote the self-similar measure associated with p_s . From the uniqueness of the invariant measure associated with the pair (Ψ, p) (see [Hut]) the measures μ_s and H^s coincide up to a constant factor. Since $\mu_s(E \setminus B_p) = 0$, then $H^s(E \setminus B_p) = 0$, and therefore $H^s(E) = H^s(B_p)$. This measure is known to be finite and positive when M is finite [Hut]. From part iv) of theorem 3 (see below) we know that $\mu_p(L_p) = 1$ for every $p \in \mathcal{P}^+$. This proves ii) in theorem 3, for the reasoning above also applies to the sets $B_{p_s}^{(\infty)}$ and L_{p_s} . \square

Since Besicovitch sets are invariants under the set mapping $S\Psi$, part iii) of theorem 3 follows from the more general result below.

Proposition 3.7 *Let $0 < t < s$, and $\phi(x) = x^t g(x)$, with g non-increasing in some interval $(0, \delta_0)$. Then, either $H^\phi(B) = 0$ or $H^\phi(B) = +\infty$, for any $S\Psi$ -invariant set B such that $\dim B > \dim(\Theta \cap B)$.*

proof:

We first prove that $H^\phi(\varphi(A)) \geq r^t H^\phi(A)$ for any similitude φ with contraction ratio $r \leq 1$, and for any $A \subset E$. Let $0 < \delta < r\delta_0$ and consider a δ -covering $\{V_i\}_i$ of $\varphi(A)$. If we write $U_i = \varphi^{-1}(V_i)$ for every i , we have

$$\sum_i \phi(|V_i|) = r^t \sum_i |U_i|^t g(r|U_i|) \geq r^t \sum_i |U_i|^t g(|U_i|) = r^t \sum_i \phi(|U_i|).$$

Since $\{U_i\}_i$ is a δ/r -covering of A , this shows that $H_\delta^\phi(\varphi(A)) \geq r^t H_{\delta/r}^\phi(A)$ and therefore the inequality $H^\phi(\varphi(A)) \geq r^t H^\phi(A)$ holds.

Assume that $0 < H^\phi(B) < +\infty$ for an $S\Psi$ -invariant set B such that $\dim B > \dim B \cap \Theta$. Then we have

$$H^\phi(B) = H^\phi(S\Psi(B)) = \sum_{i \in M} H^\phi(\varphi_i(B)) \geq \sum_{i \in M} r_i^t H^\phi(B) > H^\phi(B),$$

since $t < s$ and $\sum_{i \in M} r_i^x$ is a decreasing function of x . This contradiction implies that B has H^ϕ -measure zero or infinity. \square

Observe that proposition 3.7 holds for the gauge $\phi(\xi) = \xi^t$, $t \neq s$. This implies that there is no $S\Psi$ -invariant t -set in E for $\dim \Theta < t < s$. Notice also that proposition 3.7 applies to every $S\Psi$ -invariant subset of non-overlapping self-similar sets, to all Besicovitch-Egglestone sets, and to all *thin Besicovitch sets* since the intersection $B_p^{(\infty)} \cap \Theta$ is void (see the proof of the Θ -lemma). The result still holds for Besicovitch sets B_p such that $\dim(B_p \cap \Theta) < \dim B_p = s(p)$.

Proof of theorem 3 part iv). From part i) in theorem 3 and the definition of L_p , it follows that $\dim L_p \leq s(p)$.

We proceed as in the proof in theorem 2. Let X be the random variable in (14). For every $x \in L_p$ choose $i_x \in \pi^{-1}(x)$ such that

$$\limsup_n ((2n \log \log n)^{-1/2} S_n^X(i_x)) = d(p), \quad (26)$$

so that the ϕ_α -upper density at each $x \in L_p$ verifies

$$\bar{d}_\mu^{\phi_\alpha}(x) \geq \exp\left\{\limsup_n f_c(r_{i_x(n)}) \left(\frac{S_n^X(i_x)}{f_c(r_{i_x(n)})} - \alpha\right)\right\}. \quad (27)$$

Since $d(p) > 0$ for $p \neq p_s$, it follows from (26), lemma 3.6, and (27) that $\bar{d}_\mu^{\phi_\alpha}(x) = +\infty$ for all $0 \leq \alpha < d(p)$. The law of the iterated logarithm implies that the set L_p carries the whole measure μ_p , for B_p also does. Therefore, theorem 3.5 gives that $H^{\phi_\alpha}(L_p) = 0$ for $0 \leq \alpha < d(p)$. This gives $H^{s(p)}(L_p) = 0$, because $H^{s(p)} \leq H^{\phi_\alpha}$ for all $\alpha > 0$. \square

Closing Remark.

Some natural extensions of the research presented in this paper are to generalize to the case $M = \mathbb{N}$ the results obtained here, and to solve the same questions for the Packing dimension [Tri]. Some work addressing these issues is in progress.

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